Law of Large Numbers

MATH 354

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Intuition

- The average of results obtained from a large number of the same trials should be close to the expected value, and will tend to become closer as more trials are performed.

Example

Rolling a 6-sided die for sufficiently large number of times, we get the average value as close to 3.5, the expected value, as we want.

Weak Law of Large Numbers (WLLN)

Theorem (Weak Law of Large Numbers)

Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with expected value μ and variance $\sigma^2 < \infty$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \epsilon) = 1;$$

that is, $\bar{X}_n$ converges in probability to μ.

Definition (Convergence in Probability)

A sequence of random variables, $X_1, X_2, \ldots,$ *converges in probability* to a random variable X if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| < \epsilon) = 1.$$

Proof of WLLN

Theorem (Chebyshev's Inequality)

Let W be any random variable with mean μ and variance σ^2. For any $\epsilon > 0$,

$$P(|W - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2}.$$

- **Proof (WLLN):** According to Chebyshev's Inequality,

$$P(|\bar{X}_n - E(\bar{X}_n)| < \epsilon) \geq 1 - \frac{Var(\bar{X}_n)}{\epsilon^2}.$$

$$\text{But } E(\bar{X}_n) = E(X_i) = \mu,$$

$$Var(\bar{X}_n) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} * n\sigma^2 = \frac{\sigma^2}{n},$$

$$\text{so } P(|\bar{X}_n - \mu| < \epsilon) \geq 1 - \frac{\sigma^2}{n\epsilon^2}.$$

As $n \to \infty$, $\frac{\sigma^2}{n\epsilon^2} \to 0$. Therefore $\lim_{n\to\infty} P(|\bar{X}_n - \mu| < \epsilon) = 1.\square$

Strong Law of Large Numbers (SLLN)

Theorem (Strong Law of Large Numbers)

Let $X_1, X_2, ...,$ be independent and identically distributed random variables with expected value μ and variance $\sigma^2 < \infty$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, for every $\epsilon > 0$,

$$P(\lim_{n\to\infty} |\bar{X}_n - \mu| < \epsilon) = 1;$$

that is, $\bar{X}_n$ converges almost surely to μ.

Definition (Almost Sure Convergence)

A sequence of random variables, $X_1, X_2, ...,$ *converges almost surely* to a random variable X if, for every $\epsilon > 0$,

$$P(\lim_{n\to\infty} |X_n - X| < \epsilon) = 1.$$

SLLN is Stronger than WLLN

- SLLN is stronger than WLLN because almost sure convergence is more restrictive than convergence in probability.
- *Convergence in Probability*: $\lim_{n\to\infty} P(|X_n - X| < \epsilon) = 1$
- *Almost Sure Convergence*: $P(\lim_{n\to\infty} |X_n - X| < \epsilon) = 1$

Example

Let the sample space S be the closed interval [0,1] with the uniform probability distribution. Define the sequence $X_1, X_2, \ldots$, as follows:

$$X_1(s) = I_{[0,1]}(s), \quad X_2(s) = I_{[0,\frac{1}{2}]}(s), \quad X_3(s) = I_{(\frac{1}{2},1]}(s),$$
$$X_4(s) = I_{[0,\frac{1}{3}]}(s), \quad X_5(s) = I_{(\frac{1}{3},\frac{2}{3}]}(s), \quad X_6(s) = I_{(\frac{2}{3},1]}(s),$$

etc. Let $X(s) = 0$. Then

$$\lim_{n\to\infty} P(X_n < \epsilon) = 1 \text{ is true,}$$

$$P(\lim_{n\to\infty} X_n < \epsilon) = 1 \text{ is not true.}$$

Reference

Larsen, R. J., Marx, M. L. (2010). An introduction to mathematical statistics and its applications (5th ed.). Boston, Mass.: Prentice Hall.

Casella, G., Berger, R. L. (2002). Statistical inference (2nd ed.). (Duxbury advanced series; Duxbury advanced series in statistics and decision sciences).

- For the proof of Chebyshev's Inequality, see p.332 from Larsen and Marx.
- For the proof of Strong Law of Large Numbers, see p.268 - 269 from Casella and Berger.