Non-Monotonic Transformations of Random Variables

Nick McMullen, Daniel Ochoa
Macalester College
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1 Introduction

We know how to find the pdf from $Y = g(X)$ where g is a monotone function. What happens if g is not monotone?

2 Monotone Transformation of a Random Variable

First, for a short refresher, let's look at a monotone transformation of a random variable.

Let the random variable X have pdf $f_x(x) = \frac{30}{4}x^2(1-x)^2$ for $0 \leq x \leq 1$. Find the pdf of the random variable $Y = X^2$.

We define the random variable transformations with the information given in the following way:

$$Y = X^2 = g(x)$$
$$X = \sqrt{y} = g^{-1}(y)$$

Taking the derivative of $g^{-1}(y)$ reveals:

$$\frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y}}$$

We know that $f_y(y) = \frac{d}{dy}\ g^{-1}(y) f_x(x)$. Now, just plug in our known quantities to get:

$$\begin{aligned} f_y(y) &= \frac{1}{2\sqrt{y}} f_x(\sqrt{y}) \\ &= \frac{30}{4} y(1 - 2\sqrt{y} - y)\frac{1}{2\sqrt{y}} \end{aligned}$$

Simplification reveals:

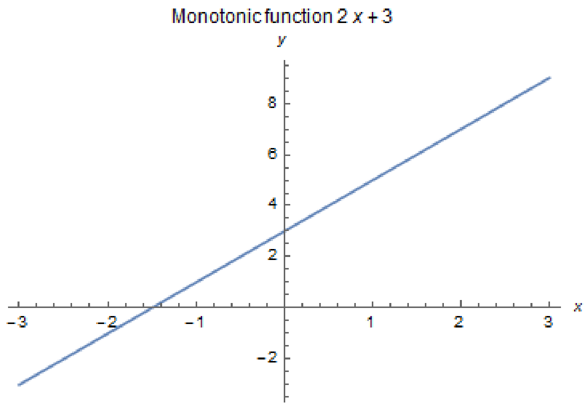
$$\begin{aligned} f_y(y) &= \frac{30}{8}\sqrt{y}(1 - 2\sqrt{y} - y) \\ &= \frac{15}{4}\sqrt{y}(1 - 2\sqrt{y} - y) \qquad (1) \end{aligned}$$

The transformation of a random variable with a monotone function amounts to calculating the inverse function g^{-1}, taking its derivative, plugging in everything to a known formula, and simplifying to get the PDF of the transformed random variable.

3 Monotonic vs. Non-monotonic

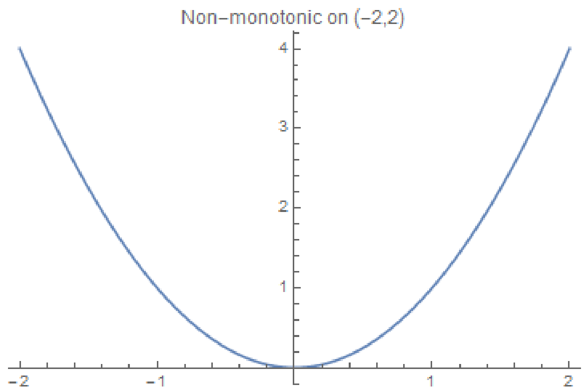
The process above seems fairly straightforward, but getting back to our original question, what if we want to transform a random variable with a non-monotone function? First, let's look at a couple of definitions to lay the foundation.

Definition: A *monotonic function* is a function whose first derivative does not change signs. Thus, it is always decreasing or always increasing, or always constant, but not more than one of these.



Definition: A *non-monotonic function* is a function whose first derivative changes signs. Thus, it is increasing or decreasing for some time and shows opposite behavior at a different location.

The quadratic function $y = x^2$ is a classic example of a simple non-monotonic function. On certain intervals this function may be monotonic, but on others, like the interval (-2,2) shown below, it is non-monotonic.



4 Non-Monotone Transformation of a Random Variable

Discrete: Suppose Jim drives along the same route every day for his 4 mile commute to work. The route begins at 0 miles (his home) and finishes at the 4 mile mark at his work. CVS Pharmacy is 4 miles in the other direction from his home, so its position is -4 miles. On any given day at a 7:45 AM, there is a 60 percent chance that he is still at home, a 10 percent chance that he is at CVS, a 10 percent chance that he is 2 miles from CVS, a 10 percent chance that he two miles from work, and a 10 percent chance that he is at work. Bored while sitting at a traffic light, Jim begins to think about what these probabilities would be like if he squared the distance, since he doesn't like negative numbers. Find the probabilities of his location using square miles, the random variable Y if we know that $Y = X^2$.

Solution: By squaring each of the discrete values in the PDF of X, $p_x(x)$, you find that there are two values at 16 (4^2 and $(-4)^2$), two values at 4 (2^2 and $(-2)^2$), and one value at 0. Adding the probabilities, you will find that Jim now has a 20 percent chance of being at 16 and a 20 percent chance of being at 4 on his new metric, while still having a 60 percent chance of being at 0. Why might this be? Because our non-monotonic transformation X^2 maps -4 and 4 to the same value (16) and the same is true for -2 and 2, which are both mapped to 4.

Continuous: In the continuous case, non-monotone transformations can be dealt with by splitting the transformation into intervals which are locally monotone.

Let's suppose X is a continuous random variable. Consider the quadratic transformation Y = X^2 for $-\infty < x < \infty$:

We can begin with an arbitrary cdf of Y:

$$\begin{aligned} F_{\mathrm{Y}}(y) = P(Y \leq y) = P(X^2 \leq y) &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_{\mathrm{X}}(\sqrt{y}) - F_{\mathrm{X}}(-\sqrt{y}) \end{aligned} \tag{2}$$

So if we wanted the pdf of the random variable Y then naturally we take the derivative of the cdf (2):

$$\begin{aligned} f_{\mathrm{Y}}(y) &= F'_{\mathrm{X}}(\sqrt{y})\left(\frac{1}{2\sqrt{y}}\right) - F'_{\mathrm{X}}(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right) \\ &= \frac{1}{2\sqrt{y}}[f_{\mathrm{X}}(\sqrt{y}) + f_{\mathrm{X}}(-\sqrt{y})], \qquad y > 0 \end{aligned} \tag{3}$$

Notice that the pdf of Y is expressed as the sum of two pieces, these pieces represent the intervals where g(x) = X^2 is monotone. In general, this will be the case.

This property can be formally defined in a theorem.

Theorem: Let X have pdf $f_X(x)$, let $Y = g(X)$. Suppose there exists a partition, $A_0, A_1, \ldots, A_K$, of X such that $P(X \in A_0) = 0$ and $f_X(x)$ is continuous on each A_i. Further, suppose there exist functions $g_1(x), g_2(x), \ldots, g_K(x)$, defined on $A_0, A_1, \ldots, A_K$, respectively, satisfying:

1. $g(x) = g_i(x)$, for $x \in A_i$,
2. $g_i(x)$ is monotone on A_i,
3. The set $Y = \{y : y = g_i(x) \; \exists x \in A_i\}$ is the same for each i = 1, ... , k, and
4. $g_i^{-1}(y)$ has a continuous derivative on Y, for each i = 1, ... , k.

Then,

$$f_Y(y) = \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \quad y \in Y \qquad (4)$$
$$= 0 \quad otherwise$$

Now let's give X a pdf so that we can solve for the pdf of Y.

Let X have the standard normal distribution with pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \; -\infty < x < \infty$$

Consider $Y = X^2$. The function $g(x) = x^2$ is monotone on $(-\infty, 0)$ and $(0, \infty)$. We should also note that the set Y = $(0, \infty)$.

We can partition X such that:

$$A_0 = \{0\}$$
$$A_1 = (-\infty, 0), \quad g_1(x) = x^2, \quad g_1^{-1}(y) = -\sqrt{y}$$
$$A_2 = (0, \infty), \quad g_2(x) = x^2, \quad g_2^{-1}(y) = \sqrt{y}$$

Thus if we employ monotone transformations on the A_1 and A_2 intervals then we can solve for the pdf of the random variable Y.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} \left| -\frac{1}{2\sqrt{y}} \right| + \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} \left| \frac{1}{2\sqrt{y}} \right|$$
$$= \frac{1}{\sqrt{2\pi y}} e^{-y/2}, \quad 0 < y < \infty \qquad (5)$$

Calculating the non-monotonic transformation of a random variable amounts to partitioning the function into its monotonic parts, calculating the transform for each of those partitions, and adding each partition. Furthermore you can see that this way to solve for the pdf of Y is equivalent to the generalization (3) where we began with an arbitrary cdf of X. If you were to integrate the pdf (5) over its support you would find that it is equivalent to 1 and thus a legitimate pdf.

Now you try. Find the pdf for the transformation of the random variable X $\sim$ N(0,1) when $Y = |X|$.

Solution: Notice that $Y = |X|$ is monotone on $(-\infty, 0)$ and $(0, \infty)$.

We can partition X such that:

$$
\begin{aligned}
&A_0 = \{0\} \\
&A_1 = (-\infty, 0), \quad g_1(x) = |x|\,, \quad g_1{}^{-1}(y) = -y \\
&A_2 = (0, \infty), \quad g_2(x) = |x|, \quad g_2{}^{-1}(y) = y
\end{aligned}
$$

Then:

$$
\begin{aligned}
f_{\mathrm{Y}}(y) &= \frac{1}{\sqrt{2\pi}} e^{-y^2/2} |-1| + \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, |1| \\
&= \frac{2}{\sqrt{2\pi}} e^{-y^2/2}, \quad 0 < y < \infty
\end{aligned} \tag{6}
$$

If you integrated the pdf (6) over its support you would find that it too is equivalent to 1.